Hybrid Ensemble Kalman Filtering and Optimal Interpolation

A new adaptive formulation

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National Center for Atmospheric Research
Data Assimilation Research Section (DAReS) - TDD - CISL

CISL Computational & Information Systems Lab

NCAR NATIONAL CENTER FOR ATMOSPHERIC RESEARCH

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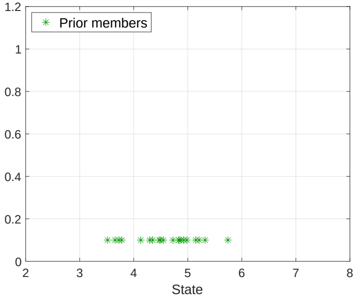

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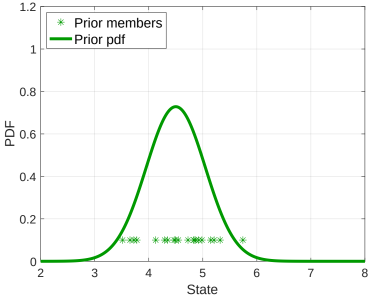

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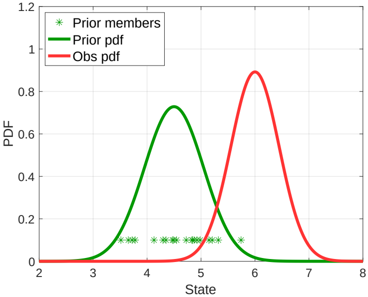

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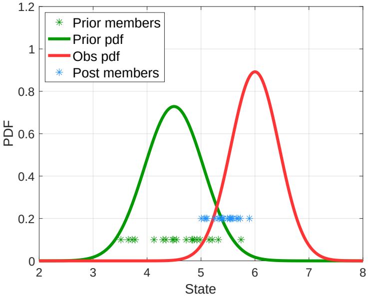

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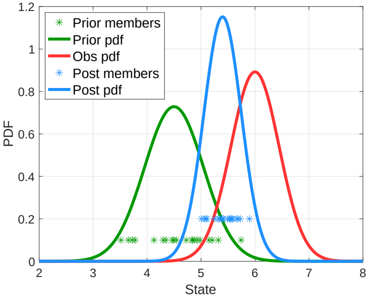

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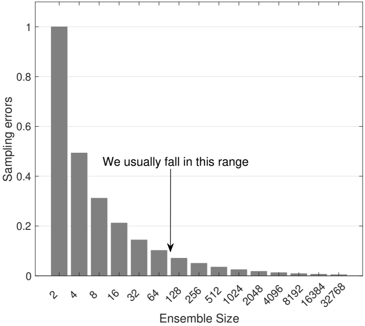

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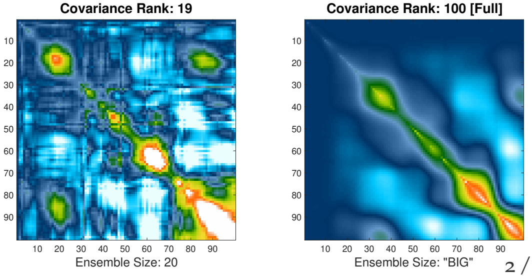

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- Others known errors:
  nonGaussianity, nonlinearity, regression errors, ...

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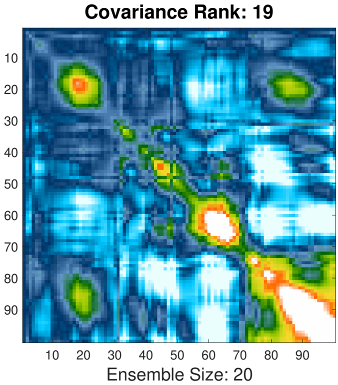

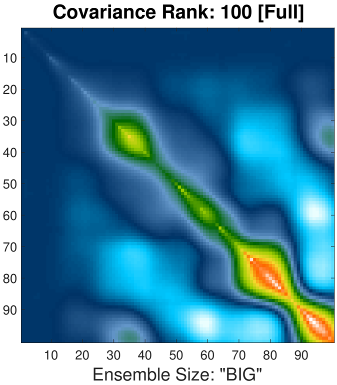

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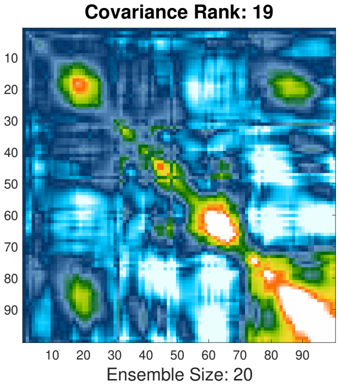

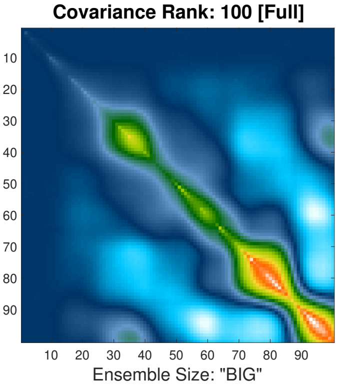

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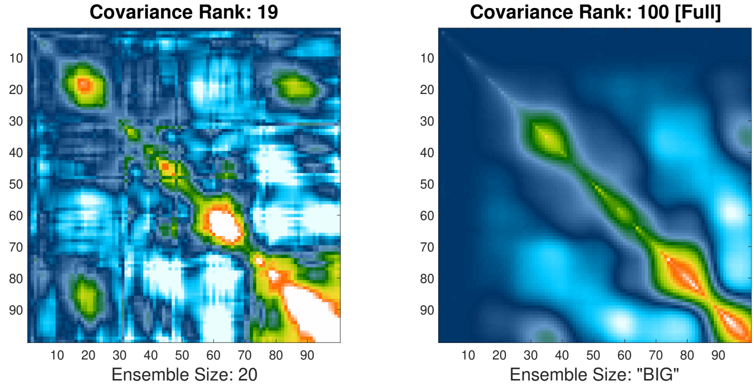

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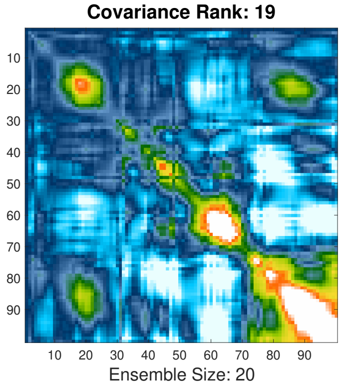

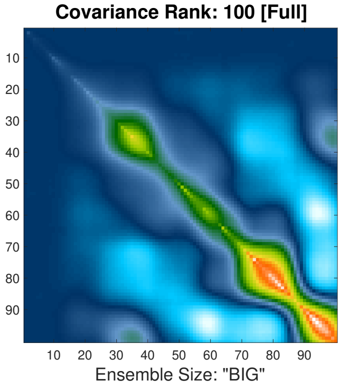

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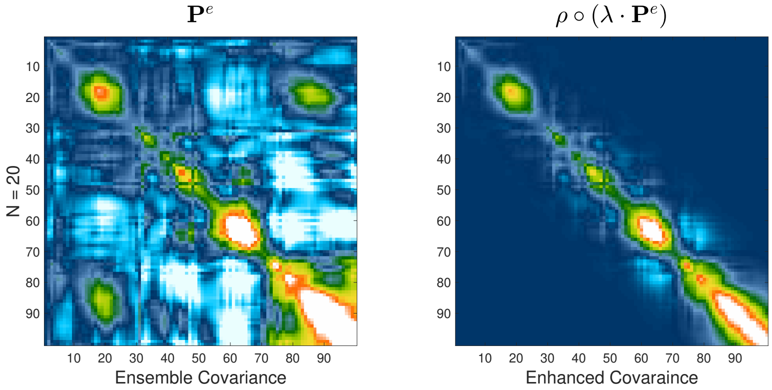

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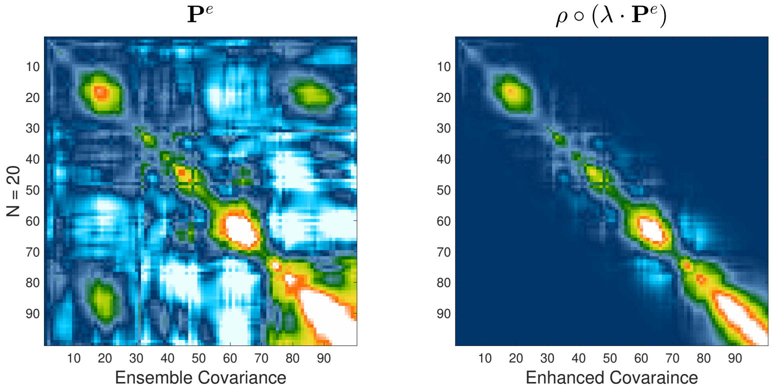

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3. **Hybridization:** $\mathbf{P}^f = \alpha \mathbf{P}^e + (1 - \alpha)\mathbf{B}$

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- OI: Optimal Interpolation (essentially a KF with a prescribed invariant $\mathbf{P}^f$)
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- Many different *hybrid* forms in the literature:
  - 4DEnVar: 4DVar with background covariance from an ensemble
  - En4DVar: Use an ensemble to approximate adjoint
  - hybrid 4(3)DVar: Var methods using a combination of climatological and ensemble covariances (e.g., α-control method in GSI)
  - EnVar: Term used for any of the previous hybrid forms

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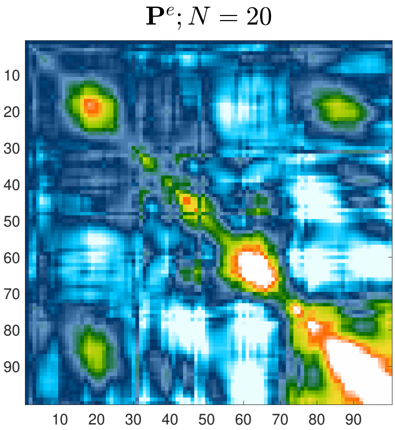

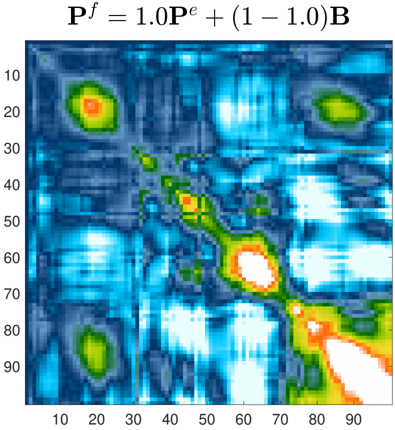

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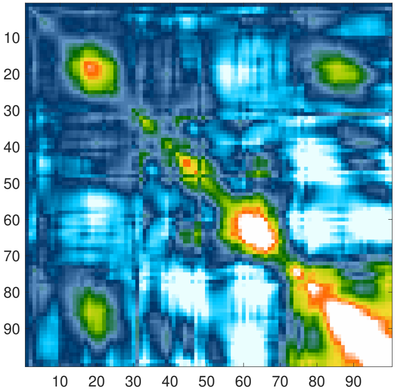
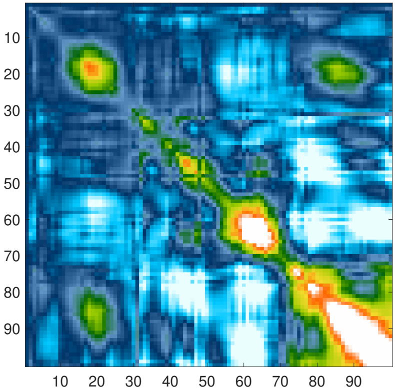


$\mathbf{P}^f = 0.9\mathbf{P}^e + (1 - 0.9)\mathbf{B}$



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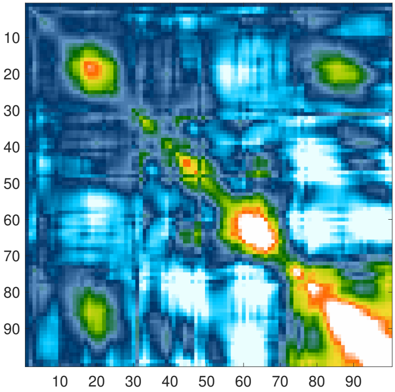
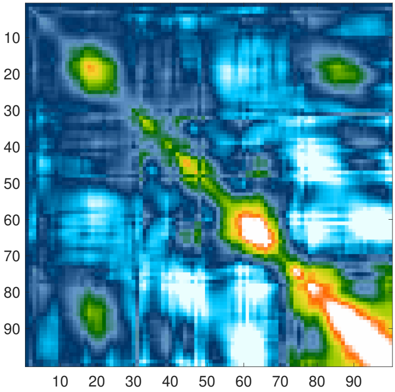


$\mathbf{P}^f = 0.8\mathbf{P}^e + (1 - 0.8)\mathbf{B}$



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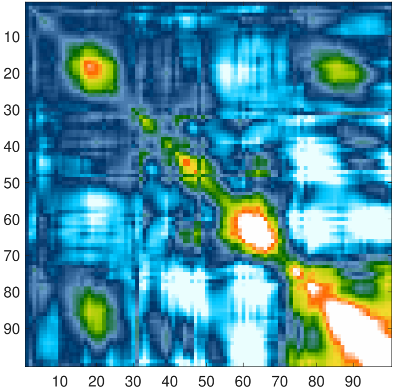
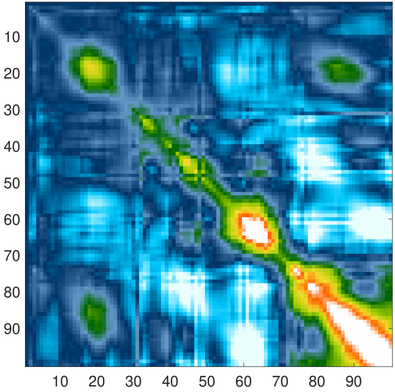


$\mathbf{P}^f = 0.7\mathbf{P}^e + (1-0.7)\mathbf{B}$



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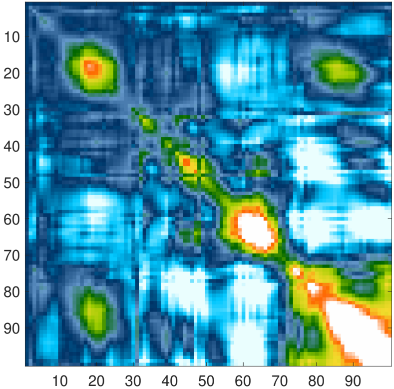
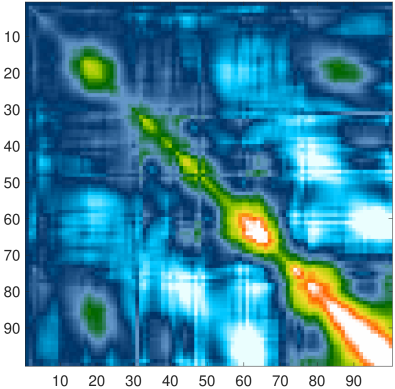


$\mathbf{P}^f = 0.6\mathbf{P}^e + (1-0.6)\mathbf{B}$



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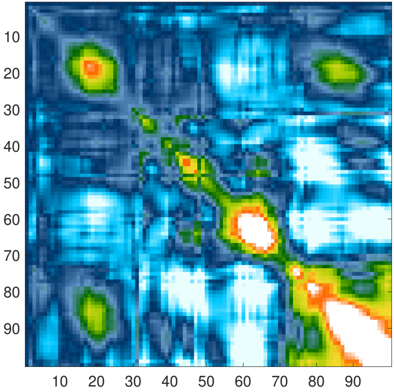

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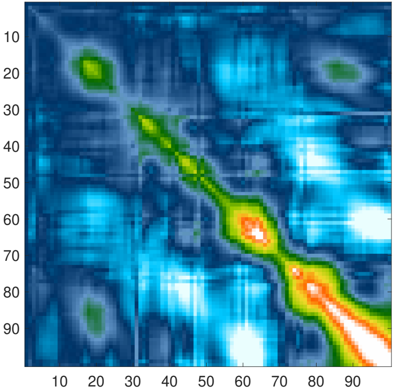

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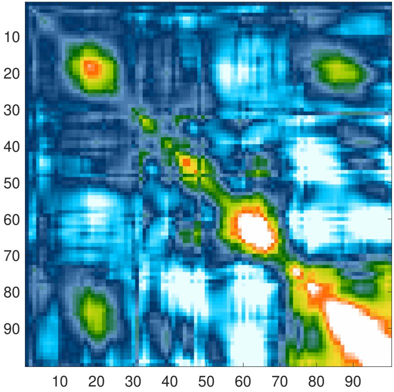

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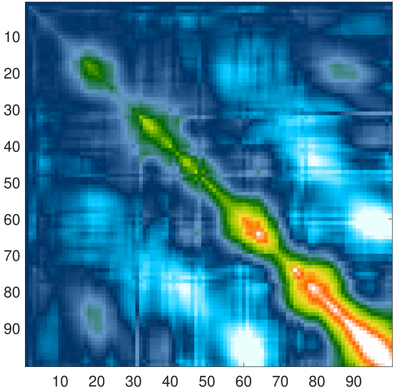

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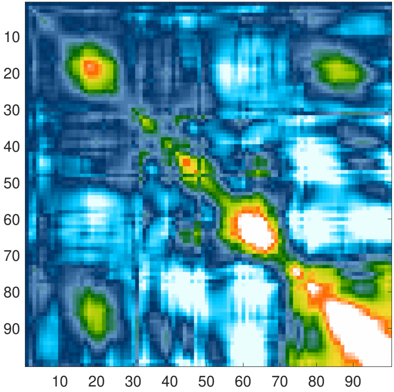

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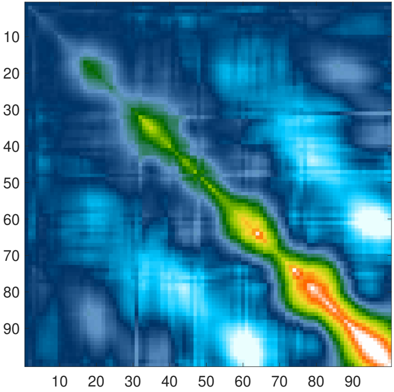

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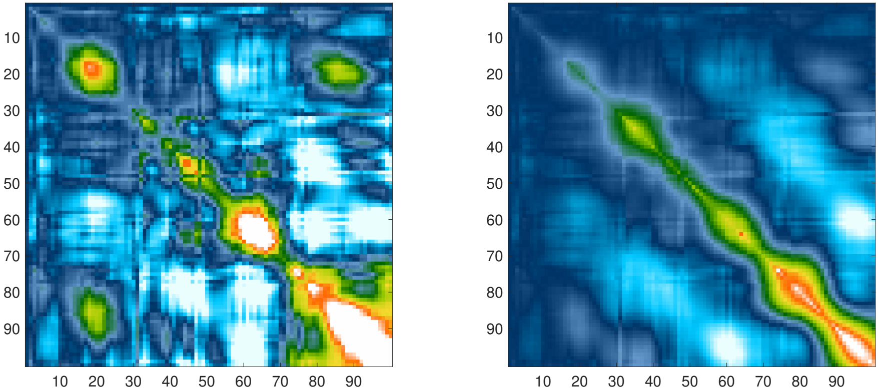
$\mathbf{P}^e; N = 20$ $\quad\quad$ $\mathbf{P}^f = 0.2\mathbf{P}^e + (1-0.2)\mathbf{B}$



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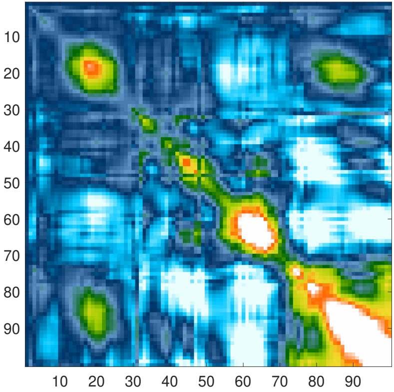
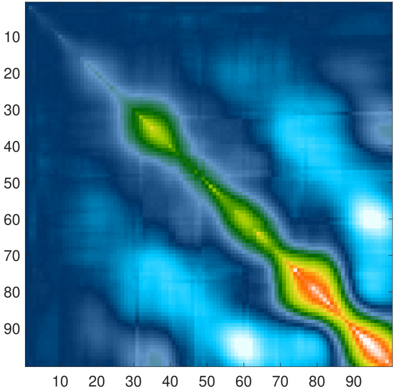


$\mathbf{P}^f = 0.1\mathbf{P}^e + (1-0.1)\mathbf{B}$



2.2 Hybrid EnKF-OI: in action

$$\mathbf{P}^f = \alpha \mathbf{P}^e + (1-\alpha)\mathbf{B}$$

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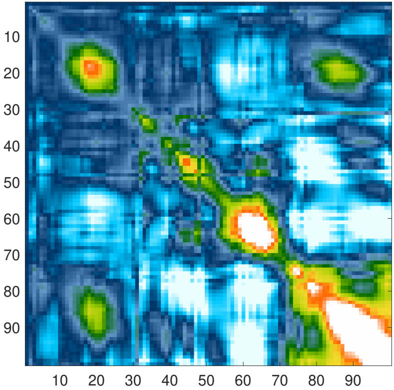
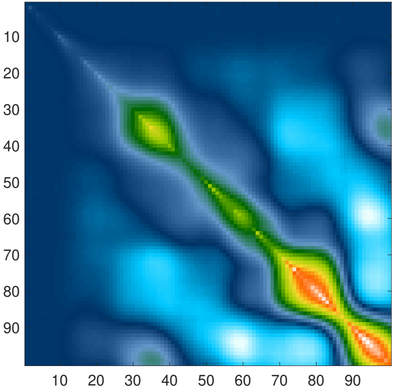


$\mathbf{P}^f = 0.0\mathbf{P}^e + (1-0.0)\mathbf{B}$



Changes to the rank, variance, correlations, norm .. of the covariance

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- Available from 3DVar systems (e.g., NMC method)
- Often formed from a large inventory of historical forecasts sampled over large windows (practical choice)

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- Storage issue: $\mathbf{B}$ is of size $(N_x \times N_x)$; N_x is the state dimension
  - Do we need to store **the entire B matrix?** May only need access to the historical (climatology) realizations

2.4 Hybrid EnKF-OI: Adaptive Form

How to choose α?

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How to choose α?

- The ensemble statistics should satisfy (Desroziers et al., 2005):

$$\mathbb{E}\left[\mathbf{d}\mathbf{d}^T\right] = \mathbf{R} + \mathbf{H}\mathbf{P}^f\mathbf{H}^T, \tag{5}$$

where $\mathbf{d} = \mathbf{y}^o - \mathbf{H}\mathbf{x}^f$. Substitute the hybrid covariance form in eq. (5):

$$\mathbb{E}\left[\mathbf{d}\mathbf{d}^T\right] = \mathbf{R} + \alpha\mathbf{H}\mathbf{P}^e\mathbf{H}^T + (1-\alpha)\mathbf{H}\mathbf{B}\mathbf{H}^T, \quad 0 \le \alpha \le 1 \tag{6}$$

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  - Assume α to be a random variable
  - Start with a prior distribution for α: $p(\alpha) \sim \mathcal{N}, \mathcal{B}, ..$
  - Use the data to construct a likelihood function: $p(\mathbf{d}|\alpha)$
  - Use Bayes' rule to find an updated estimate of α:

$$p(\alpha|\mathbf{d}) \approx p(\alpha) \cdot p(\mathbf{d}|\alpha) \tag{7}$$

  - Posterior α can be used as the prior for the next DA cycle

2.4 Hybrid EnKF-OI: Adaptive Form cont.

```
switch Prior
    case 'Gaussian'
```

$$p(\alpha) = \mathcal{N}\left(\alpha_f, \sigma_{\alpha_f}\right) \equiv \frac{1}{\sqrt{2\pi\sigma^2_{\alpha_f}}} \exp\left[-\frac{\left(\alpha - \alpha_f\right)^2}{2\sigma^2_{\alpha_f}}\right]$$

```
    case 'Beta'
```

$$p(\alpha) = \mathcal{B}(\gamma, \beta) \equiv \alpha^{\gamma-1}(1-\alpha)^{\beta-1}\frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)}$$

```
end
```

2.4 Hybrid EnKF-OI: Adaptive Form cont.

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$$p(\alpha) = \mathcal{B}(\gamma, \beta) \equiv \alpha^{\gamma-1}(1-\alpha)^{\beta-1}\frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)}$$

```
end
```

Likelihood:

$$\begin{aligned} \theta(\alpha) &= \text{trace}(\mathbf{R}) + \alpha\,\text{trace}\left(\mathbf{H}\mathbf{P}^e\mathbf{H}^T\right) + (1-\alpha)\,\text{trace}\left(\mathbf{H}\mathbf{B}\mathbf{H}^T\right) \\ p(\mathbf{d}|\alpha) &= \frac{1}{\sqrt{2\pi\theta(\alpha)}} \exp\left[-\frac{\mathbf{d}^T\mathbf{d}}{2\theta(\alpha)}\right] \end{aligned}$$

Posterior: $p(\alpha|\mathbf{d})$ is either near Gaussian or near Beta

2.5 Hybrid EnKF-OI: Illustration

Scalar Example
6 parameters

$\mathbf{P}^e$	$\sigma_e^2 = 0.9$
$\mathbf{B}$	$\sigma_s^2 = 0.2$
$\mathbf{R}$	$\sigma_o^2 = 0.1$
$\mathbf{d}$	$d = 2.5$

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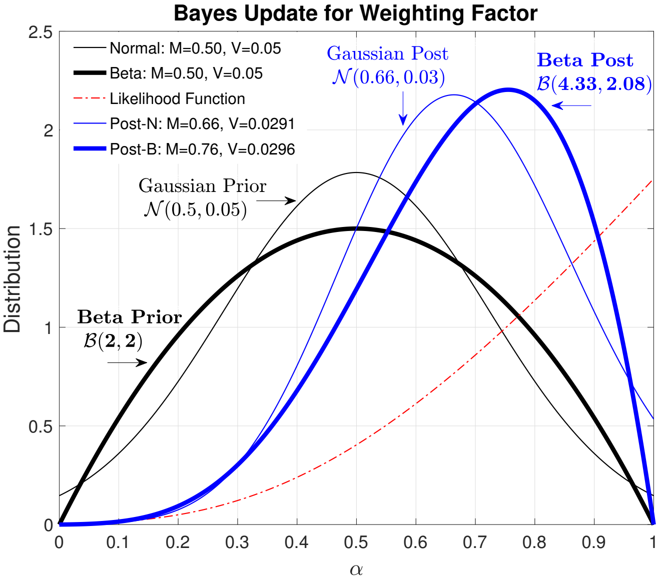

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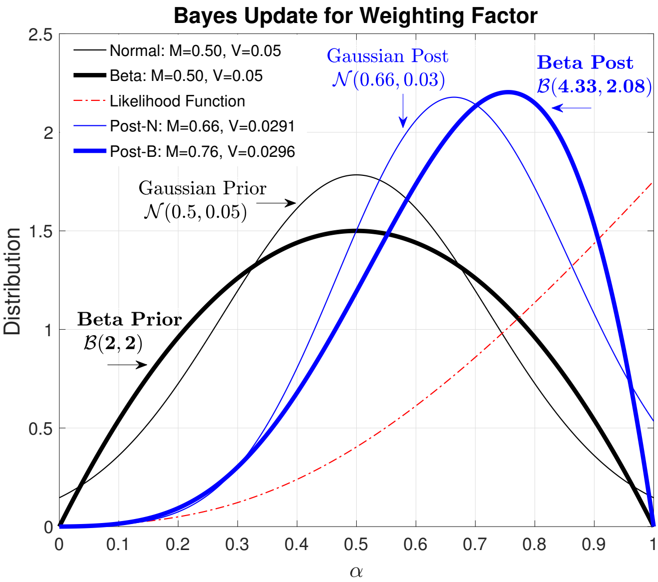
Large bias causes α to increase (i.e., larger weight given to σ_e^2)



2.6 Hybrid EnKF-OI: Implementation

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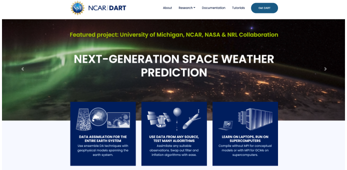
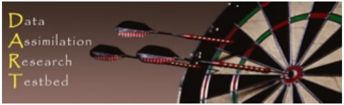
dart.ucar.edu
docs.dart.ucar.edu





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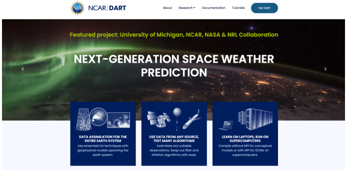
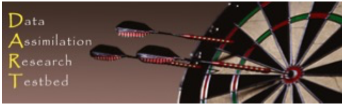
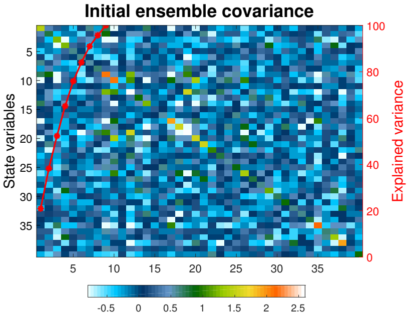


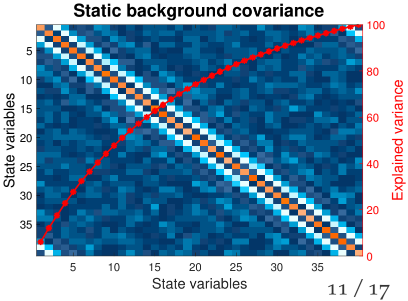


- Can assume the hybrid weight to be spatially-varying
  - Biases are not homogenous in space
  - Heterogenous observation networks (densely observed regions tend to have small ensemble spread)
  - Need to assimilate the observations serially

3.1 Experiments using L96

- L96: 40 variables
- Observe every other variable ($\mathbf{R} = 1$)
- Observe every 5 time steps
- **B** Climatological run (1000)
- $p(\alpha) \sim \mathcal{N}(0.5, 0.1)$





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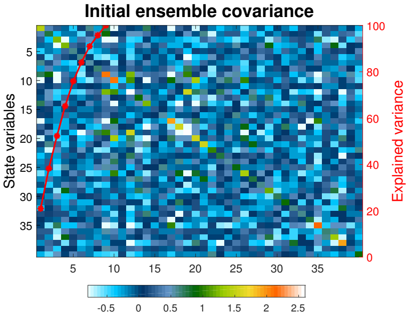
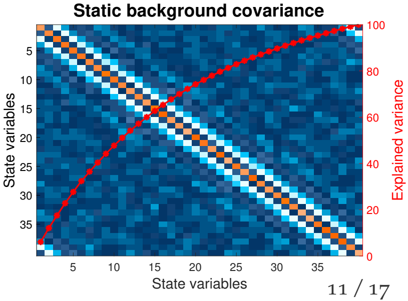
Sensitivity Tests [1]
Perfect OSSEs

- Ensemble size
- Obs. Network

Sensitivity Tests [2]
Model Errors

- Inflation
- Localization





3.2 Sensitivity Tests: Ensemble Size

1. **EnKF**
2. **EnOI**: EnKF with fixed **B** (Hybrid; $\alpha = 0$)
3. **EnKF-OI**; $\alpha = 0.5$
4. **AC-EnKF-OI**: Adaptive, spatially-Constant EnKF-OI
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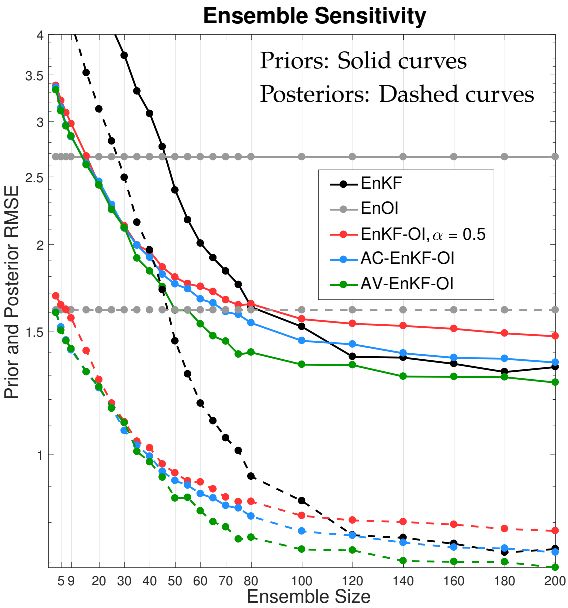

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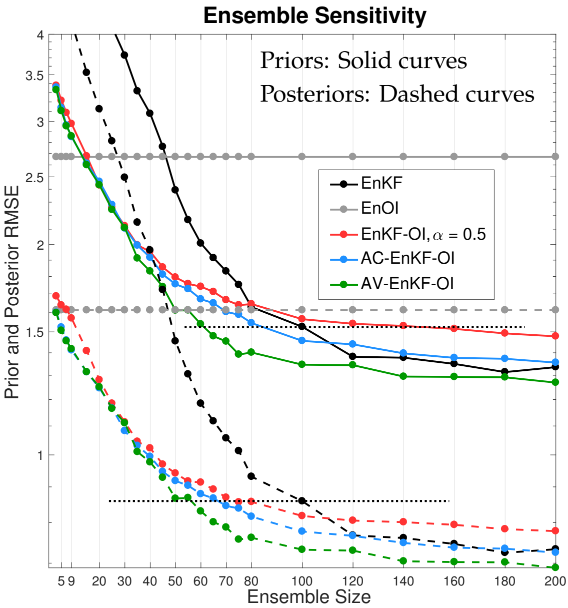
★ **EnKF**'s accuracy is reproduced by the hybrid schemes with 40 – 50% less ensemble members



3.2 Sensitivity Tests: Ensemble Size

- **AC-EnKF-OI**: Dashed lines
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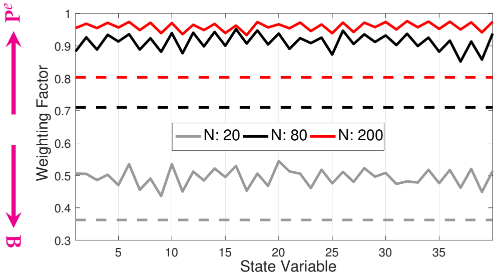

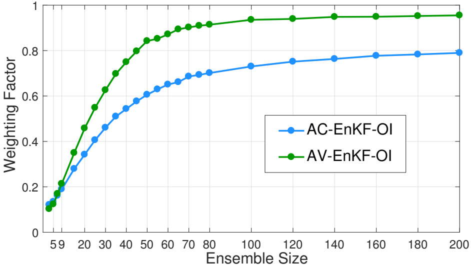

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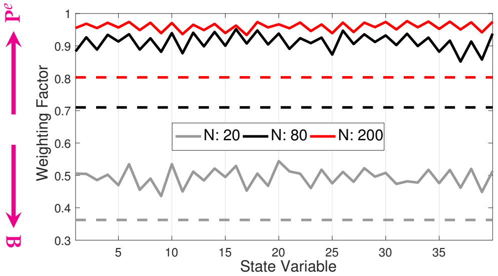
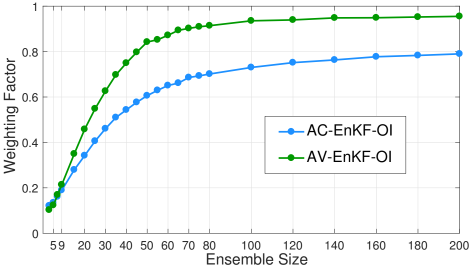


- For small ensembles, both adaptive spatially-constant and varying schemes behave the same
- AV-EnKF-OI responds more efficiently to changes in the ensemble



3.3 Sensitivity Tests: Observation Network

- **Data Void I:** Observe the first 20 variables
- **Data Void II:** Observe the first and last 5 variables
- **Data Void III:** Observe 10 center variables
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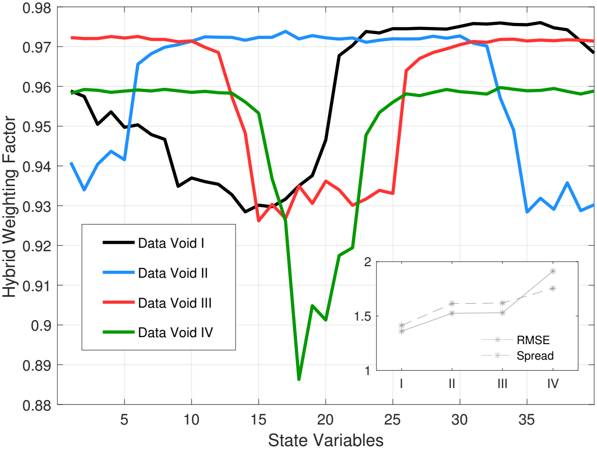

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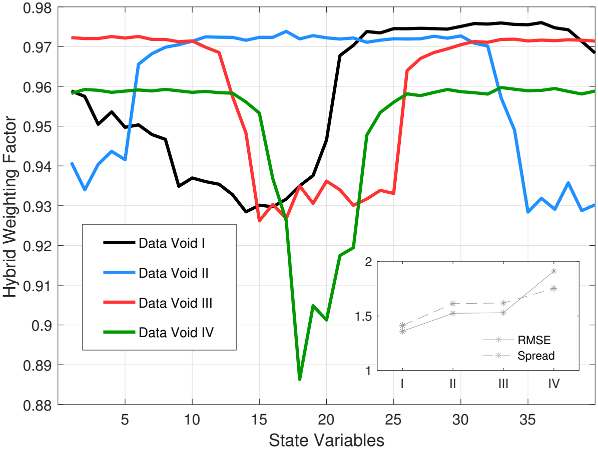
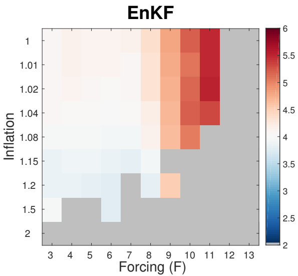
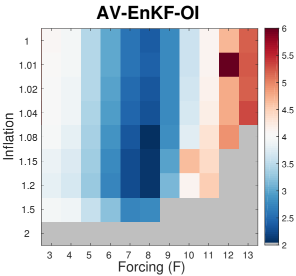


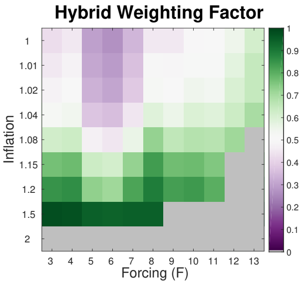
- In densely observed regions, the ensemble spread decreases
- Hybrid scheme places weight more on **B** to increase the variance, allowing better data fit

Sensitivity Tests: Model Errors + Inflation

- Ensemble size: 20
- Model error; vary $3 \leq F \leq 13$
- $\mathbf{B}$ is generated in each case using biased F
- No localization







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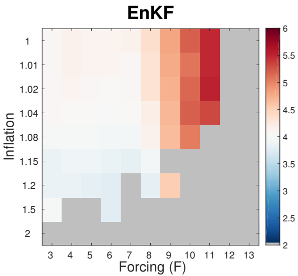
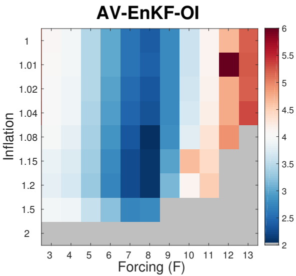
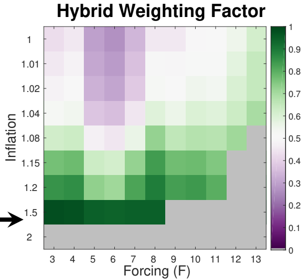




- Hybrid scheme: better stability and more accurate even in very biased conditions
- As inflation increases, adaptive α increases (more weight on the ensemble covariance)



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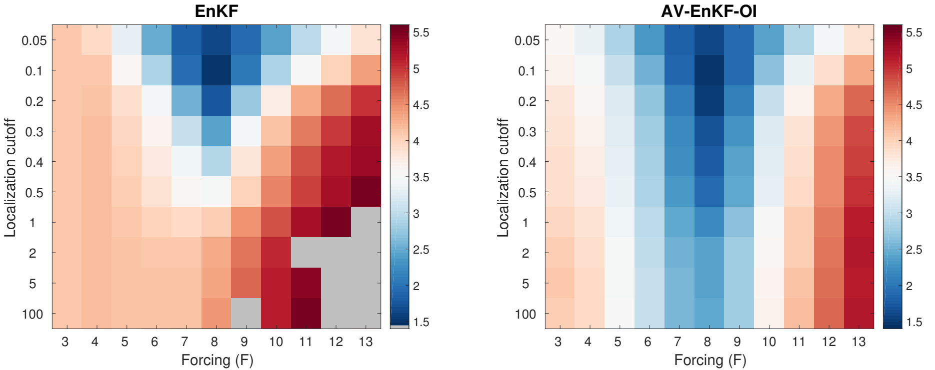

- $N_e = 20$, No inflation
- Vary both F and localization length scale
- Adaptive hybrid scheme is systematically better than the EnKF for all tested cases

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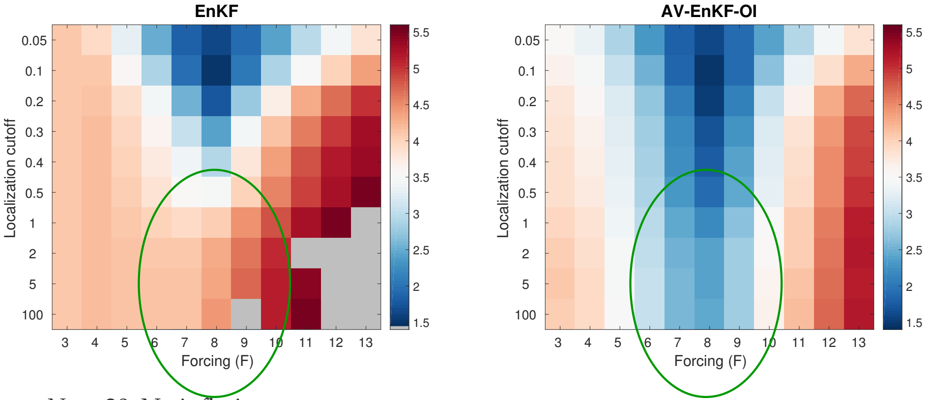
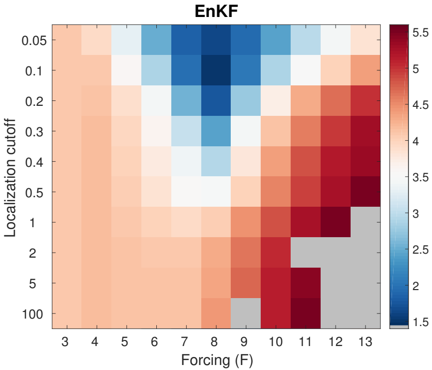
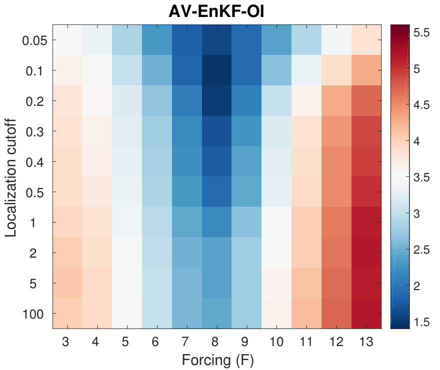
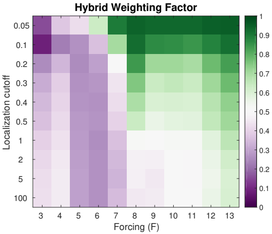


- $N_e = 20$, No inflation
- Vary both F and localization length scale
- Adaptive hybrid scheme is systematically better than the EnKF for all tested cases
- With very little to no localization, hybrid scheme still performs exceptionally well
- Does the climatological flavor from **B** mitigate spurious correlations?

Sensitivity Tests: Model Errors + Localization







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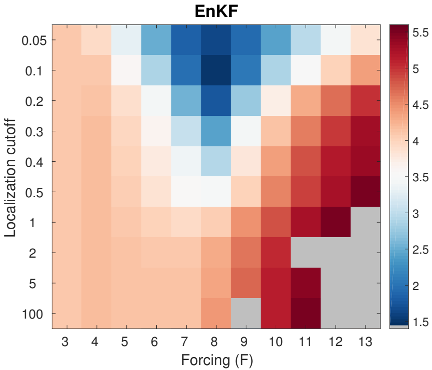
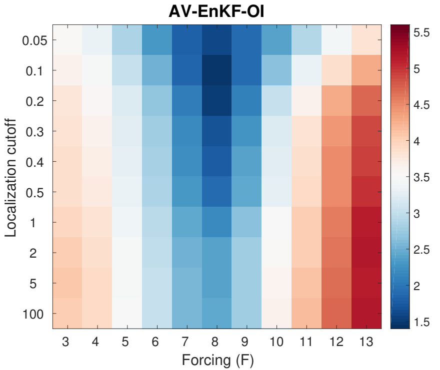
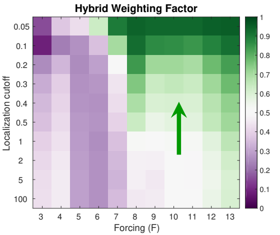




- For chaotic behaviour (i.e., $F \geq 8$): As localization increases, α increases



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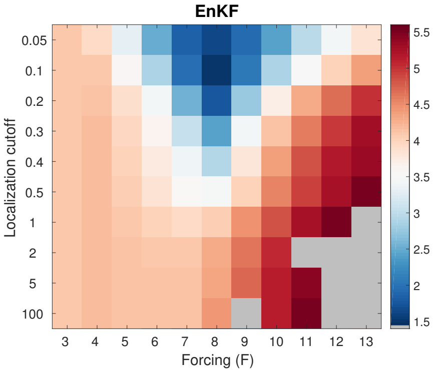
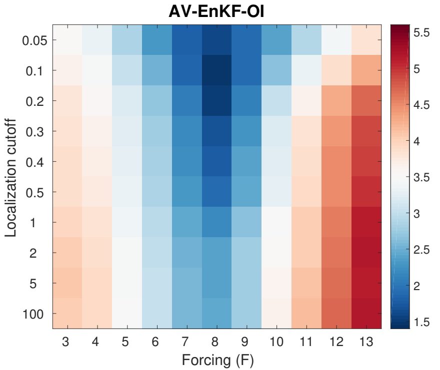
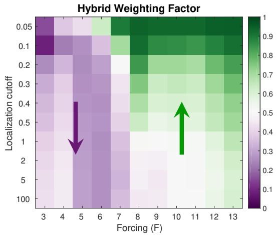




- For chaotic behaviour (i.e., $F \geq 8$): As localization increases, α increases
- Less chaotic (smaller ensemble variance): α decreases to *bring-in* variability from **B**



4.1 Concluding Remarks

- Prior (background) ensemble covariance **must** be enhanced
- On top of inflation and localization, hybridizing $\mathbf{P}^e$ with stationary OI-based background covariances can be helpful and perhaps crucial
- The adaptive scheme uses available data through Bayes rule to determine the relative weighting between the ensemble and the static covariance
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– Only flow-dependent covariance	– OI flavor & flow-dependent information
– Requires a large ensemble size	– Works well with fairly small ensembles
– Fair computational cost	– Storage, additional IO cost
– Strong tuning (inf, loc, ..)	– Fully adaptive, requires less inf, loc, ..
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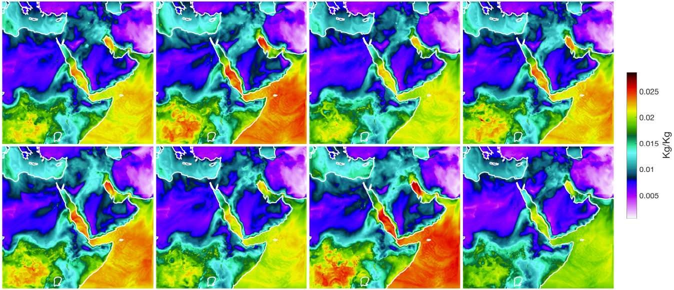
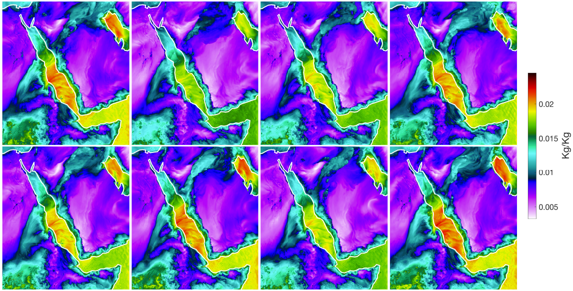
El Gharamti, M. (2021). Hybrid Ensemble-Variational Filter: A Spatially and Temporally Varying Adaptive Algorithm to Estimate Relative Weighting. Monthly Weather Review, 149(1), 65-76.

4.2 Future: Applications to Earth System Models





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